Efficiently computing annuity conversion factors via feed-forward neural networks

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Sascha Günther, ETH Zürich

(joint work with Peter Hieber (University of Lausanne)

and Maria Aragona (University Torino))

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The Setting

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We have an economic scenario generator with risk factors $\mathcal{Z}$:

- interest rate risk
- longevity risk
- inflation
- stock market,
- ...

An insurance company guarantees an **annuity conversion factor** at retirement (in $T \in \mathbb{N}$ years from now)

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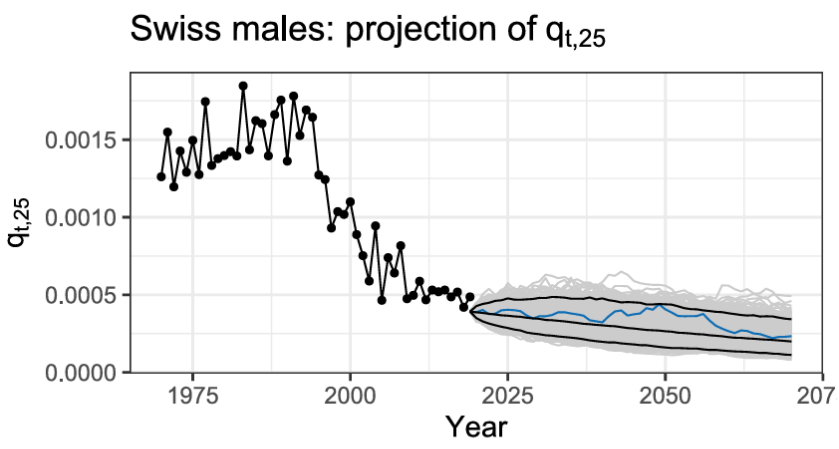
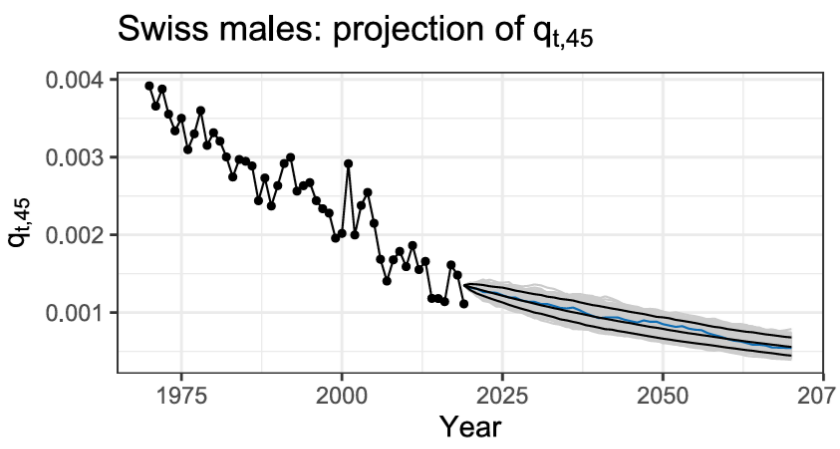
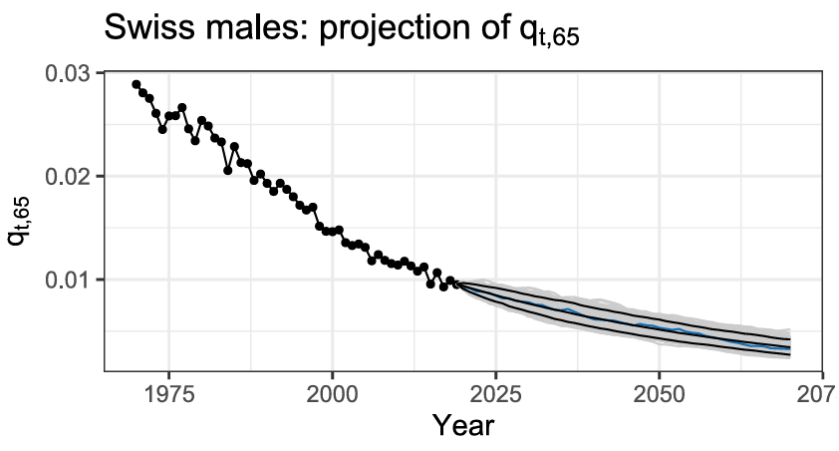
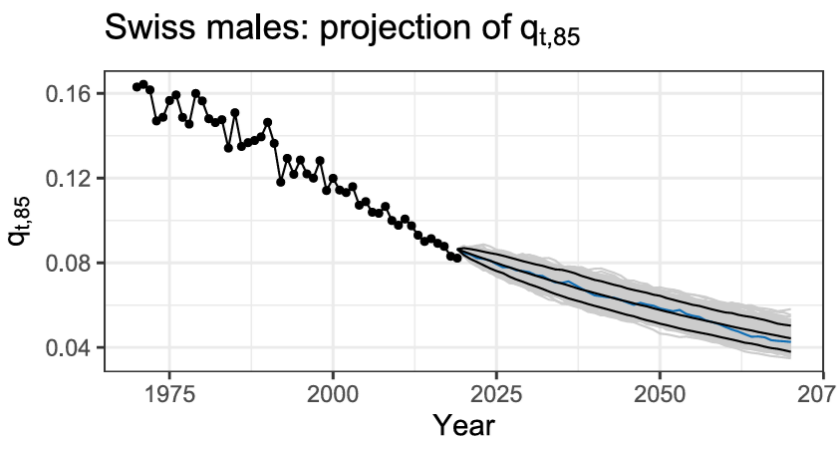
Mortality risk









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Example: 4 risk factors

- For **longevity risk**, use **2-factor** Li-Lee model (see Li and Lee (2005)) with factors k_t and K_T

- For **interest rate risk** use **2-factor** Hull-White / G2++ model (see also Graf and Korn (2020), Günther and Hieber (2024))

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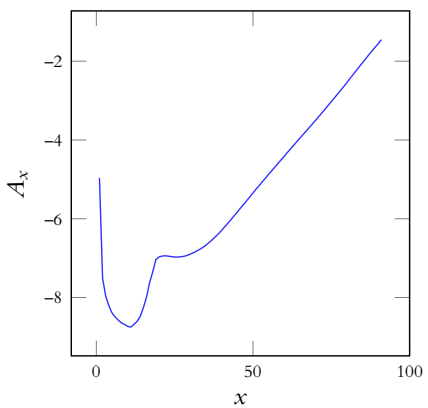
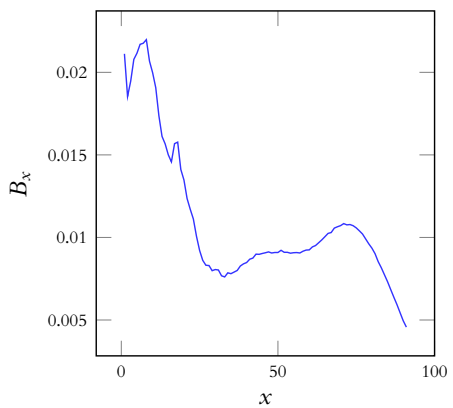
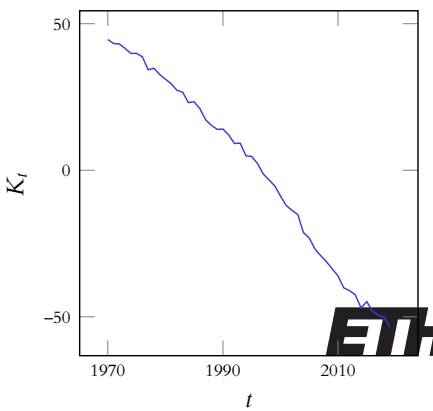
$$\ln(m_{x,t}) = \ln(m^{\mathrm{g}}_{x,t}) + \ln(m^{\mathrm{c}}_{x,t})\,,$$
$$\ln(m^{\mathrm{g}}_{x,t}) = A_x + B_x K_t\,,$$
$$\ln(m^{\mathrm{c}}_{x,t}) = a_x + b_x k_t\,,$$
$$K_t = K_{t-1} + \gamma + \epsilon_t\,,$$
$$k_t = c + d \cdot k_{t-1} + \delta_t\,.$$







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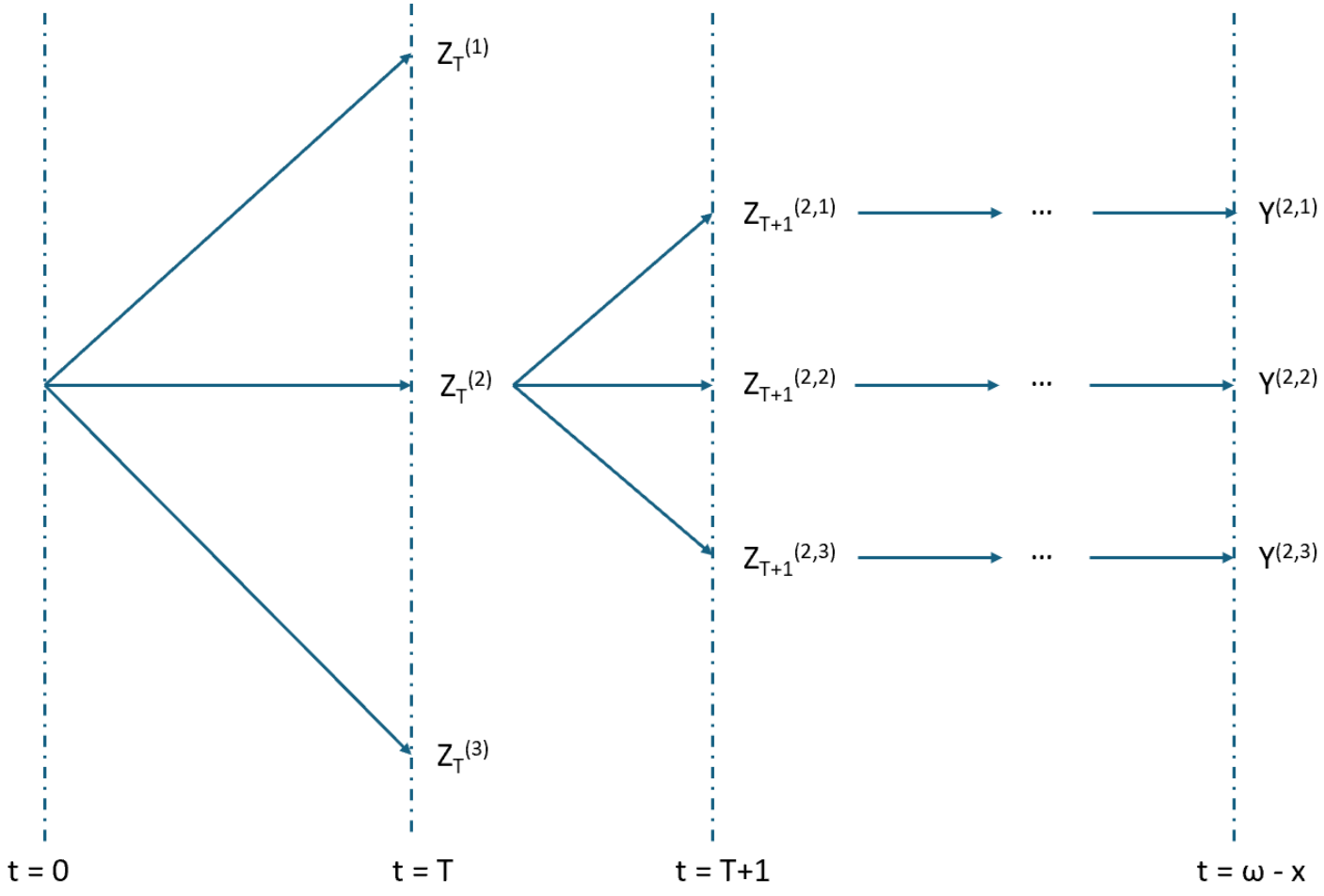
$$
\begin{aligned}
r_t &= \psi_t + x_t + y_t\,, \\
\mathrm{d}x_t &= -a\,x_t\,\mathrm{d}t + \nu\,\mathrm{d}W_t^{(1)}\,, \\
\mathrm{d}y_t &= -b\,y_t\,\mathrm{d}t + \eta\Big(\rho_r\,\mathrm{d}W_t^{(1)} + \sqrt{1-\rho_r^2}\,\mathrm{d}W_t^{(2)}\Big)\,, \\
\psi_t &= f^M(0,t) + \text{volatility correction}\,,
\end{aligned}
$$

where $f^M(0,t)$ is a Nelson-Siegel-Svensson yield curve.

Nested simulations



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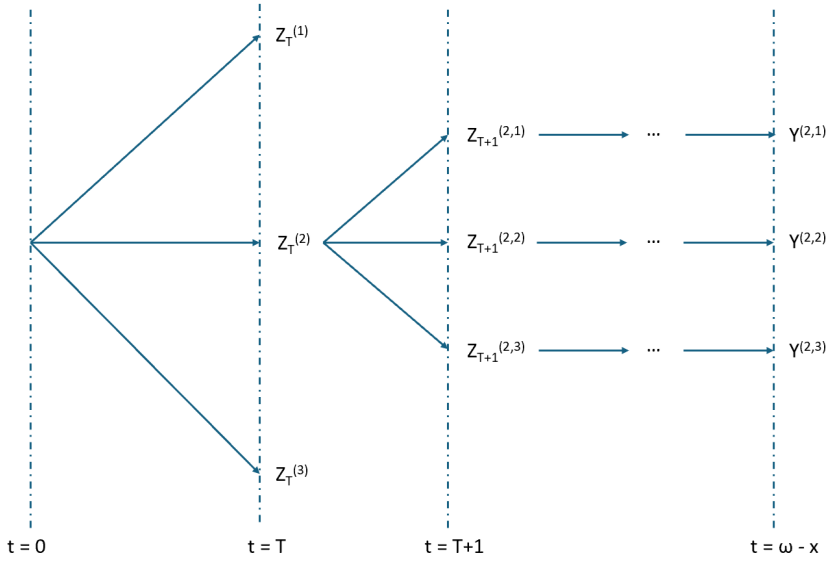

$$\begin{aligned}\text{Number of simulation steps} &= \text{outer scenarios} \times \text{inner scenarios} \times \text{time steps} \\ &= 1\,000\,000 \times 1\,000 \times 60 = 6 \cdot 10^{10} \\ \text{time taken} &= \text{Number of steps} \times \text{time taken per step} \\ &= 6 \cdot 10^{10} \cdot 1\,\text{ms} \approx 1.9 \text{ years}\end{aligned}$$

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Inner estimation

- Risk factors are stored in the (random) vector $\{\boldsymbol{Z}_t\}_{t\geq 0}$
  e.g. $\boldsymbol{Z}_t = (K_t, k_t, x_t, y_t)$

- We need to **efficiently evaluate** the annuity conversion factor **conditional on** $\boldsymbol{Z}_T = \boldsymbol{z}$:

$$a_x(T, \boldsymbol{z}) = \mathbb{E}\left[\sum_{t=1}^{\omega-x-T} \mathrm{e}^{-\int_T^{T+t} r_s \,\mathrm{d}s} \mathrm{e}^{-\sum_{s=0}^{t-1} m_{x+T+s,T+s}} \,\middle|\, \boldsymbol{Z}_T = \boldsymbol{z} \right].$$

There is no closed-form solution for this expectation!

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Literature review

Literature (selection)

- **linear least-square regression** of inner simulations based on underlying risk factors:
  Longstaff and Schwartz (2001), Bauer et al., 2010, Floryszczak et al., 2016, Ha and Bauer, 2022, Bacinello et al., 2024

- **neural network** regressions:
  Gan (2013), Hejazi and Jackson (2017), Cheridito et al. (2020), Fernandez-Arjona and Filipović (2022), Barigou and Delong (2022)

- approaches using **Taylor expansions**:
  Biffis and Millossovich (2006), Dowd et al. (2011), Cairns (2011)

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The articles closest to our research are:

- **Large datsets with many risk factors:** Castellani et al. (2021): internal model with approx. 20 factors,
  Krah et al. (2020), Jonen et al. (2023): 12 - 13 risk factors.
  Comparison of different techniques. Neural networks (should) show their strength.
- **Smaller models in actuarial science:** Bacinello et al. (2021) apply least-squares Monte-Carlo to annuity conversion factors.
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Our approach

Our approach

- "Interpolate" between feed-forward neural networks and least-squares Monte-Carlo:
  - use the (unusual) **polynomial activation** function $\varphi(x) = x^2$.
  - work with **small networks** (1 layer, $\approx$ 4-20 neurons)
  - Starting point: least-squares Monte-Carlo as **special case** of neural networks

Least-squares Monte-Carlo (LSMC)

- We choose:

$$a_x(T, \boldsymbol{z}) \approx e(\boldsymbol{z}) = \sum_{i=1}^{M} \beta_i e_i(\boldsymbol{z}) \,,$$

  where $e_i(\boldsymbol{z})$ are polynomial basis functions up to degree 2, e.g. $e_i(\boldsymbol{z}) = z_1^2$ or $e_i(\boldsymbol{z}) = z_2 \cdot z_3$.

- The weights β_i are determined according to a least-squares regression:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \,.$$

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Feed-Forward Neural Networks (NN)

We choose a **single-layer** neural network with activation functions φ_k, $k = 1, 2, \ldots, K$ in the hidden layer. The **final output** is thus

$$e(\boldsymbol{z}) = \gamma_0 + \sum_{k=1}^{K} \gamma_k \cdot \varphi_k \left(\alpha_{0,k} + \sum_{i=1}^{p} \alpha_{i,k} \cdot z_i \right) ,$$

where $\alpha_{i,k}$ is the i-th weight in the k-th neuron.

First choice: Exponential linear unit

$$\varphi_k(x) = \mathrm{Elu}(x) = \begin{cases} x, & \text{if } x \geq 0, \\ a(\mathrm{e}^x - 1), & \text{if } x < 0\,. \end{cases}$$

Second choice: Squared activation function

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Numerical Results

Error measurement

- We determine the **90th percentile** $q_{90\%}^{\text{LSMC/NN}}$ of the annuity values for NN and LSMC.
- We compare the two procedures to the *true value* considering the **mean absolute percentage error** (MAPE):

$$\text{MAPE}^{\text{LSMC/NN}} = \frac{1}{100} \sum_{i=1}^{100} \left| \frac{q_{90\%}^{(i),\text{LSMC/NN}} - q_{90\%}^{\text{true}}}{q_{90\%}^{\text{true}}} \right| .$$

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Setup for LSMC and NN

- For the two regression techniques, we rely on a **small nested simulation** of 1000 outer and 10 inner simulations.
- We then do a **regression** on the $1000 \cdot 10 = 10\,000$ realisations.
- Then we **estimate** the quantile $q_{90\%}^{\mathrm{LSMC/NN}}$ on an **additional set** of $1\,000\,000$ outer scenarios.

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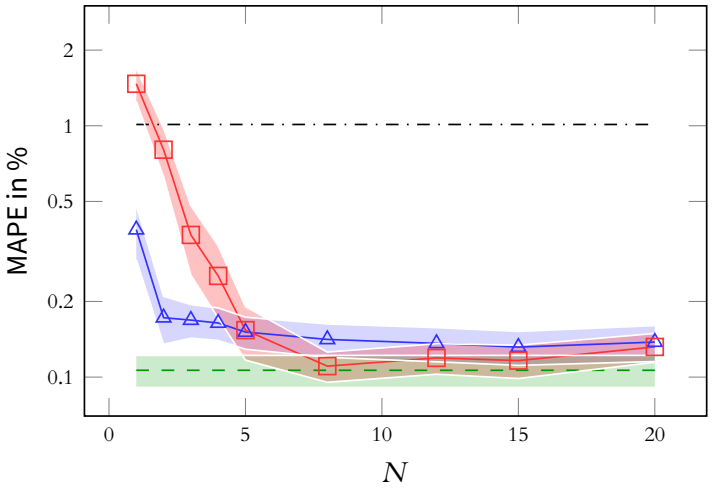
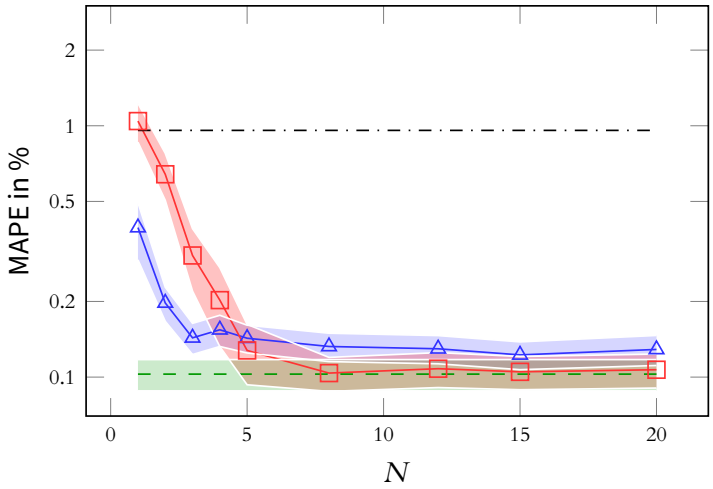
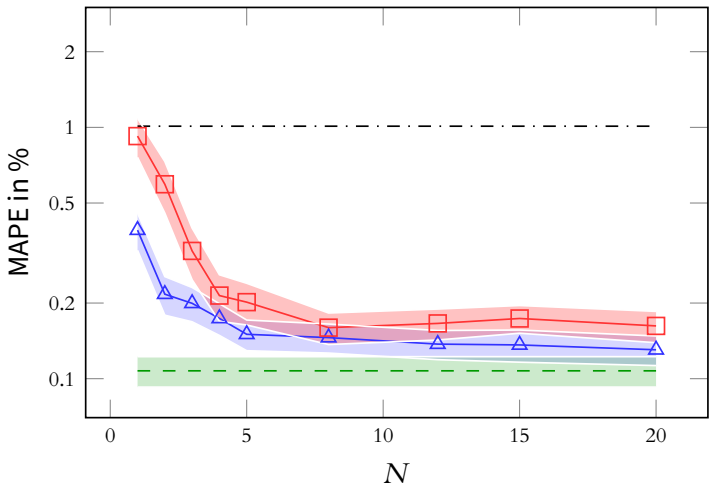


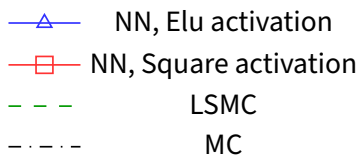
(a) loss function: Mean Squared Error



(b) loss function: Huber



(c) loss function: Mean Absolute Percentage Error



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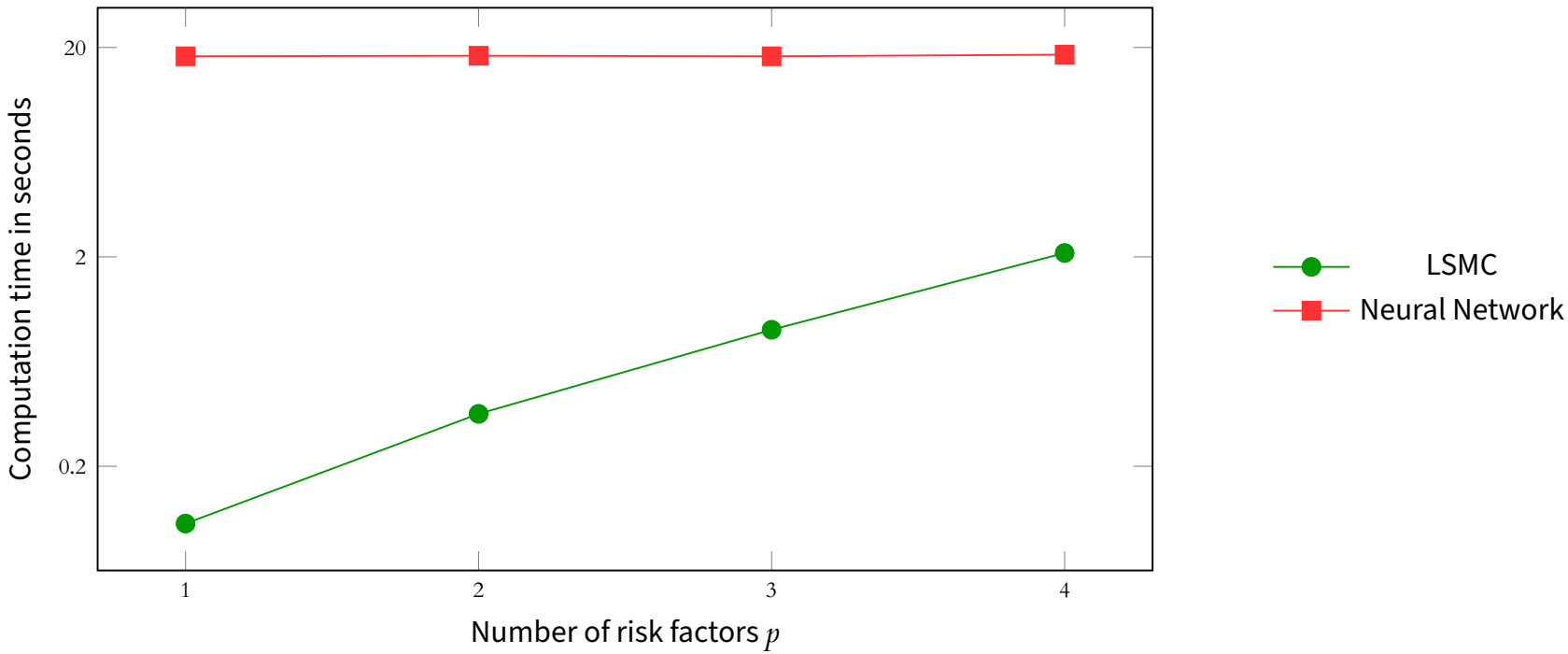
Comparison NN and LSMC

e.g. Castellani et al. (2021): We need three-layer networks with many neurons to slightly beat LSMC. LSMC performs very well.

- We confirm this result.
- We provide small networks that lead to a similar performance as LSMC, but cannot beat it.
- The link between NN and LSMC may be a starting point to improve LSMC.

Curse of dimensionality



Summary

Takeaways

- A simple polynomial **LSMC is difficult to beat** by neural networks (in this application)
- We link LSMC and neural networks, allowing a slight extension of LSMC, still keeping some interpretability (small neural networks). We interpolate between LSMC and NN - **LSMC is a special case of NN**.
- Neural networks do not suffer from the **curse of dimensionality** in terms of the number of risk factors. LSMC requires a selection of relevant basis functions / interaction terms.
- Combination of **squared activation** function and **Huber loss** works best.

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