Statistical Sciences
UNIVERSITY OF TORONTO

Modification and combination of dynamic models

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joint works with

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Motivation: Stress testing insurance portfolio

Insurance portfolio modelled via a compound Poisson process X on $[0, T]$:

- intensity $\kappa = 5$
- severity $\Gamma(2, 1)$ (expectation 1, variance 2)
- solvency horizon $T = 1$

Stress testing/believes/views:

- What if $\mathrm{VaR}_{0.5}(X_T)$ is/should be 15% larger?

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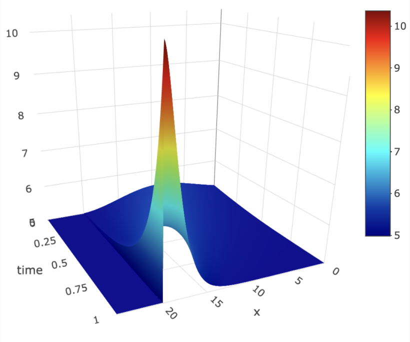
$\rightarrow$ How can we modify our model to incorporate the views?

Motivation: Stress testing insurance portfolio

Stress testing: What if $\mathrm{VaR}_{0.5}(X_T)$ is 15% larger?
Increase from 17.4 to 20.

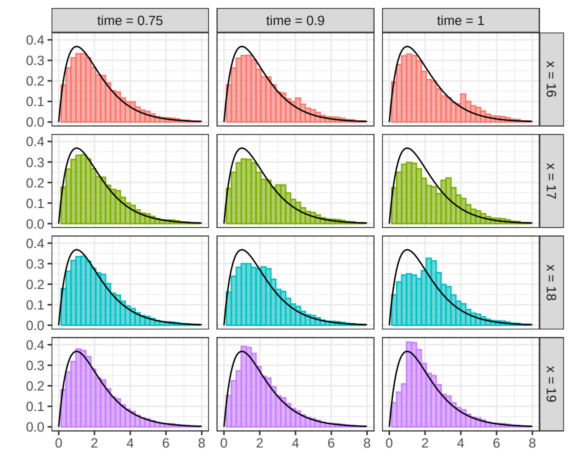
- stochastic intensity $\kappa(t, x)$



Motivation: Stress testing insurance portfolio

Stress testing: What if $\text{VaR}_{0.5}(X_T)$ is 15% larger?

- severity distribution dependent on time t and state x



Model setup

Completed filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]})$ and consider n-dim. stochastic process $(\boldsymbol{X}_t)_{t\in[0,T]}$.

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for $f_1, \ldots, f_{r_1}, g_1, \ldots, g_{r_2} : \mathbb{R}^n \to \mathbb{R}$

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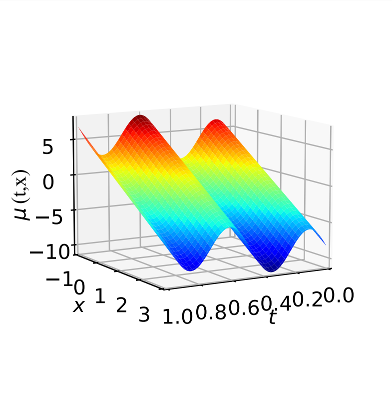
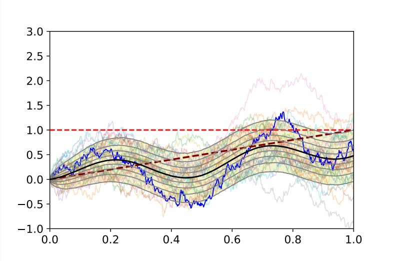
  for $f_1, \dots, f_{r_1}, g_1, \dots, g_{r_2} : \mathbb{R}^n \to \mathbb{R}$

- What are the **dynamics** of $\boldsymbol{X}$ under $\mathbb{Q}$ that is **“closest”** to $\mathbb{P}$?

Motivation

$\mathbb{P}$

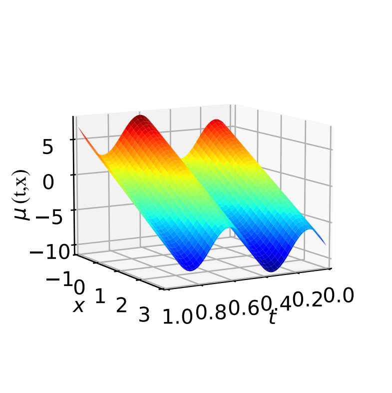


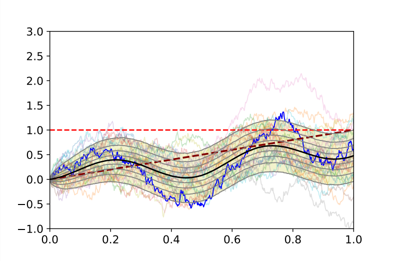


$\mathbb{E}[X_1] \approx 0.483$ and $\mathbb{E}[\int_0^1 \mathbb{1}_{\{X_t<t\}}\, dt] \approx 0.627$

Motivation

$\cancel{\mathbb{P}} \to \mathbb{Q}$ but 'close' to $\mathbb{P}$

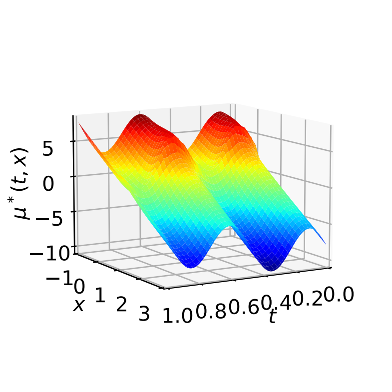


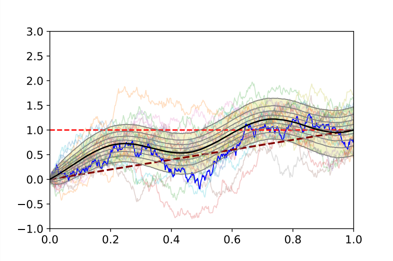


$\mathbb{E}^{\mathbb{Q}}[X_1] \approx \cancel{0.483} \;\to\; 1$ and $\mathbb{E}^{\mathbb{Q}}[\int_0^1 \mathbb{1}_{\{X_t<t\}}\, dt] \approx \cancel{0.627} \;\to\; 0.2$

Motivation

$\mathbb{Q}^*$





$\mathbb{E}^{\mathbb{Q}}[X_1] \approx\ 1$ and $\mathbb{E}^{\mathbb{Q}}[\int_0^1 \mathbb{1}_{\{X_t<t\}}\,dt] \approx 0.2$

Model Setup: m-dim. Lévy-Itô Process

$(\boldsymbol{X}_t)_{t\in[0,T]}$ satisfies the SDE

$$\mathrm{d}\boldsymbol{X}_t = \underbrace{\boldsymbol{\mu}(t,\boldsymbol{X}_t)}_{\text{drift}}\,\mathrm{d}t + \underbrace{\boldsymbol{\sigma}(t,\boldsymbol{X}_t)}_{\text{volatility}}\,\mathrm{d}\boldsymbol{W}_t + \int_{\mathbb{R}^l}\underbrace{\boldsymbol{\gamma}(t,\boldsymbol{X}_{t^-},\boldsymbol{z})}_{\text{jumps}}\,\tilde{\boldsymbol{\kappa}}(\mathrm{d}t,\mathrm{d}\boldsymbol{z})$$

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- $\boldsymbol{W}$ m-dim. $\mathbb{P}$-Brownian motion.
- $\bar{\mu}$ l-indep. Poisson random measures (PRM)
- $\boldsymbol{\nu}$ compensator of $\bar{\mu}$
- $\tilde{\boldsymbol{\kappa}} = \bar{\mu} - \boldsymbol{\nu}$ the compensated measure

$+$ assumptions.

Optimisation Problem

The KL-divergence of $\mathbb{Q}$ with respect to $\mathbb{P}$ is

$$D_{KL}\left(\mathbb{Q} \,||\, \mathbb{P}\right) := \mathbb{E}\left[\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \log\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right)\right].$$

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We consider the optimisation problem

$$\inf_{\mathbb{Q}\in\boldsymbol{\mathcal{Q}}} D_{KL}(\mathbb{Q} \,||\, \mathbb{P}) \quad \text{s.t.}$$

$$\mathbb{E}^{\mathbb{Q}}\left[f_j(\boldsymbol{X}_T)\right] = 0\,, \;\; j = 1, 2, \ldots, r_1\,, \quad \text{and} \tag{P}$$

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T g_j(\boldsymbol{X}_s)\,\mathrm{d}s\right] = 0\,, \;\; j = 1, 2, \ldots, r_2\,,$$

where $\boldsymbol{\mathcal{Q}}$ is the set of equivalent probability measures...

Theorem 1 (Verification)

Define $J^\dagger(t,\boldsymbol{x}) := -\log \omega^\dagger(t,\boldsymbol{x})$*, where*

$$\omega^\dagger(t,\boldsymbol{x}) = \mathbb{E}_{t,\boldsymbol{x}}\left[\exp\left(-\boldsymbol{\eta}_1 \cdot \boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{\eta}_2 \cdot \int_t^T \boldsymbol{g}(\boldsymbol{X}_u)\mathrm{d}u\right)\right]$$

Let

$$\boldsymbol{\lambda}_t^\dagger := -\frac{\nabla_{\boldsymbol{x}}\omega^\dagger(t,\boldsymbol{X}_t)}{\omega^\dagger(t,\boldsymbol{X}_t)}\,\boldsymbol{\sigma}(t,\boldsymbol{X}_t) \qquad \textit{and}$$

$$\boldsymbol{h}_t^\dagger(\boldsymbol{z}) := -\frac{\boldsymbol{\Delta}_{\boldsymbol{z}}\omega^\dagger(t,\boldsymbol{X}_{t-})}{\omega^\dagger(t,\boldsymbol{X}_{t-})}\,,$$

then, under suitable assumptions, $\boldsymbol{\lambda}^\dagger$ *and* $\boldsymbol{h}^\dagger$ *are admissible controls and* $J^\dagger = \inf_{\mathbb{Q}_{\boldsymbol{\lambda},\boldsymbol{h}} \in \boldsymbol{\mathcal{Q}}} J^{\boldsymbol{\lambda},\boldsymbol{h}}$.

Representation of optimal RN density

Theorem 2 (Optimal Measure Change)

Let Lagrange multipliers $(\boldsymbol{\eta}_1^*, \boldsymbol{\eta}_2^*)$ *solve*

$$\nabla_a \, \log\left(\mathbb{E}\left[e^{-a\cdot\left(\boldsymbol{f}(\boldsymbol{X}_T),\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\,\mathrm{d}u\right)}\right]\right)\Big|_{a=(\boldsymbol{\eta}_1^*,\boldsymbol{\eta}_2^*)} = 0\,.$$

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Then, the optimal measure $\mathbb{Q}^*$ *has RN-density*

$$\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{P}} = \frac{\exp\left(-\boldsymbol{\eta}_1^*\cdot\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{\eta}_2^*\cdot\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\,\mathrm{d}u\right)}{\mathbb{E}\left[\exp\left(-\boldsymbol{\eta}_1^*\cdot\boldsymbol{f}(\boldsymbol{X}_T) - \boldsymbol{\eta}_2^*\cdot\int_0^T \boldsymbol{g}(\boldsymbol{X}_u)\,\mathrm{d}u\right)\right]}\,,$$

- Insurance company with 2 LoB:
  modelled by bivariate compound Poisson process $(X_t^1, X_t^2)_{t\in[0,T]}$.

What if: Stress testing an insurance portfolio

Kroell, P., Jaimungal IME '23

- Insurance company with 2 LoB:
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- Risk measure of interest $\mathrm{VaR}_{0.9}\big(X_T^1 + X_T^2\big)$

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! Stress on first LoB at $T/2$: $\mathrm{VaR}_\alpha(X_{T/2}^1)$

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? How large a stress to achieve a 5% increase in $\mathrm{VaR}_{\mathbf{0.9}}\big(\boldsymbol{X_T^1 + X_T^2}\big)$ **?**

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α	% increase in $\mathrm{VaR}_{0.9}(X_{T/2}^1)$
0.5	30.4
0.6	24.8
0.7	19.6
0.8	16.1

Compound Poisson process with intensity $\kappa = 5$, marginal severities $\xi_1 \overset{\mathbb{P}}{\sim} \Gamma(2,1)$, $\xi_2 \overset{\mathbb{P}}{\sim} \mathrm{Exp}(2)$ with a t copula with correlation 0.8 and 3 degrees of freedom.

Improving models through barycentres

Experts' models

- Expert $k \in \{1, \ldots, K\}$ has a probability measure $\mathbb{P}_k$ and thinks that the m-dim. process $(\boldsymbol{X}_t)_{t\in[0,T]}$ satisfies the SDE

$$\mathrm{d}\boldsymbol{X}_t = \underbrace{\mu^{(k)}(t, \boldsymbol{X}_t)}_{\text{drift}}\,\mathrm{d}t + \underbrace{\sigma(t, \boldsymbol{X}_t)}_{\text{volatility}}\,\mathrm{d}\boldsymbol{W}_t^{(k)}$$

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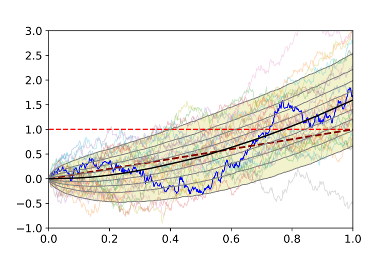
$\Rightarrow$ How to combine the expert models?

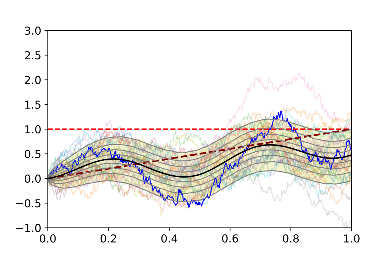
$\Rightarrow$ How to simultaneously incorporate beliefs/views?

Experts' models

model - I



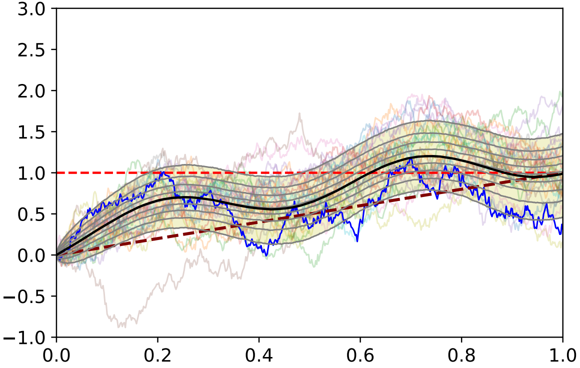
model - II



Example

$$\pi_1 = 0.1$$

$$\mathbb{E}^{\mathbb{Q}}[X_1] = 1 \quad \& \quad \mathbb{E}^{\mathbb{Q}}[\int_0^1 \mathbb{1}_{\{X_t \leq t\}}] = 0.2$$



Example

$$\pi_1 = 0.3$$

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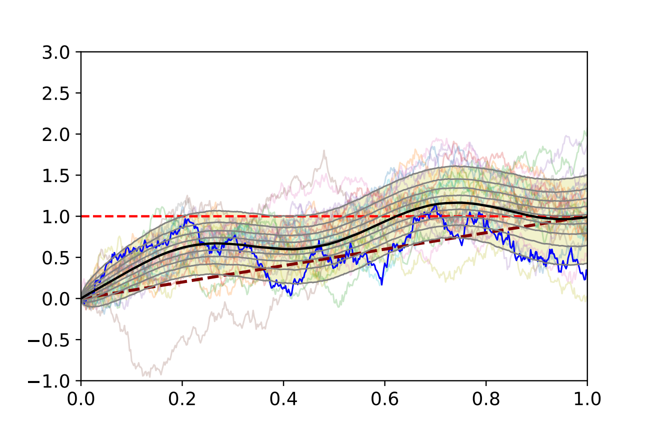

Example

$$\pi_1 = 0.5$$

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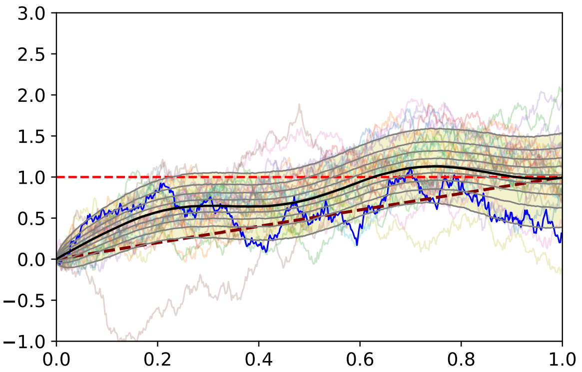

Example

$$\pi_1 = 0.7$$

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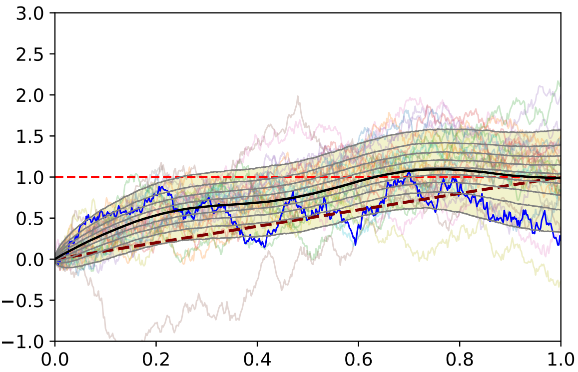

Example

$$\pi_1 = 0.9$$

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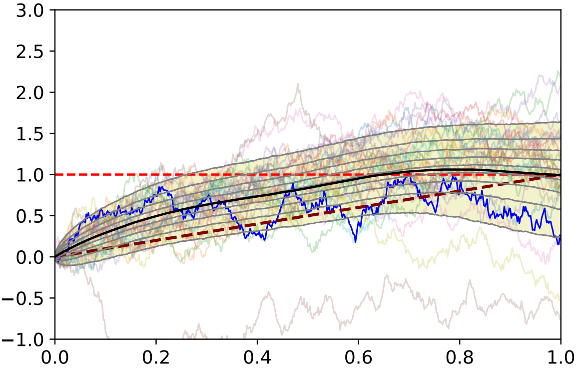

Simple model averaging

- The **average drift measure** $\mathbb{Q}[\bar{\boldsymbol{\mu}}]$: $\boldsymbol{X}$ has drift

$$\bar{\boldsymbol{\mu}}(t,x) := \sum_{k \in \mathcal{K}} \pi_k \, \mu^{(k)}(t,x)\,,$$

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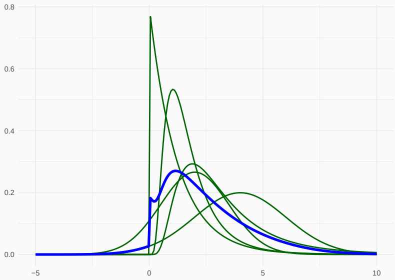
- combining random variable: six different expert densities



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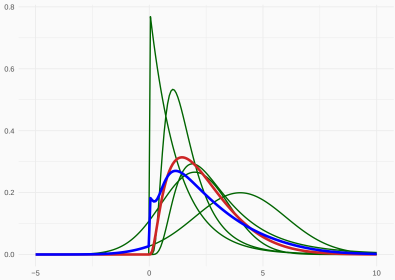
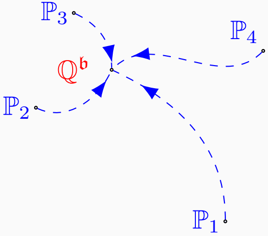


Kullback-Leibler barycentre

Find the model that is the

Barycentre of models

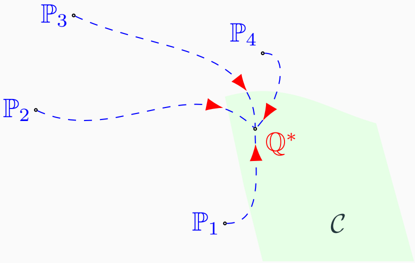
$$\arg\min_{\mathbb{Q}\in\boldsymbol{\mathcal{Q}}} \sum_{k=1}^{K} \pi_k \, D_{KL}\left(\mathbb{Q} \;||\; \mathbb{P}_k\right)$$



Kullback-Leibler barycentre

Find the model that is the constrained Barycentre of models

$$\arg\min_{\mathbb{Q}\in\boldsymbol{\mathcal{Q}}} \sum_{k=1}^{K} \pi_k \, D_{KL}\left(\mathbb{Q} \,||\, \mathbb{P}_k\right) , \quad \text{subject to constraints on } \boldsymbol{X}_T \text{ under } \mathbb{Q}$$



Experts compatibility

Assumption 3 (Model compatibility)

$$\mathbb{E}^{\mathbb{Q}[\bar{\boldsymbol{\mu}}]}\left[e^{-\int_0^T \varsigma(u,X_u)\,du}\right] < +\infty\,,$$

where $\Sigma(t,x) = \sigma(t,x)^{\intercal}\sigma(t,x)$ *and*

$$\varsigma(t,x) := \sum_{k\in\mathcal{K}} \pi_k \left(\mu^{(k)}(t,x) - \bar{\boldsymbol{\mu}}(t,x)\right)^{\intercal} \Sigma^{-1}(t,x) \left(\mu^{(k)} - \bar{\boldsymbol{\mu}}(t,x)\right)\,.$$

– akin to a weighted Mahalanobis distance

Pure barycentre

Proposition 4 (Pure barycentre)

Under some technical assumptions, the barycentre measure exists, is unique, and its RN derivative has representation

$$\frac{\mathrm{d}\mathbb{Q}^{\mathfrak{b}}}{\mathrm{d}\mathbb{Q}[\bar{\boldsymbol{\mu}}]} = \frac{e^{-\int_0^T \varsigma(t,X_t)\,\mathrm{d}t}}{\mathbb{E}^{\mathbb{Q}[\bar{\mu}]}\left[e^{-\int_0^T \varsigma(t,X_t)\,\mathrm{d}t}\right]}\;.$$

Constraint barycentre

Theorem 5

Let Lagrange multipliers (η_0^*, η_1^*) *solve*

$$\nabla_\eta \log\left(\mathbb{E}^{\mathbb{Q}^\flat}\left[e^{-\eta_0 \int_0^T g_t \, \mathrm{d}t - \eta_1 f_T}\right]\right) = 0\,.$$

Constraint barycentre

Theorem 5

Let Lagrange multipliers (η_0^*, η_1^*) *solve*

$$\nabla_\eta \log\left(\mathbb{E}^{\mathbb{Q}^{\mathfrak{b}}}\left[e^{-\eta_0 \int_0^T g_t \,\mathrm{d}t - \eta_1 f_T}\right]\right) = 0\,.$$

Then, the optimal measure $\mathbb{Q}[\theta_{\eta^*}]$ *has RN-density*

$$\frac{d\mathbb{Q}[\theta_{\eta^*}]}{d\mathbb{Q}[\bar{\mu}]} = \frac{e^{-\int_0^T (\varsigma(t,X_t)+\eta_0^* g_t)\,dt - \eta_1^* f_T)}}{\mathbb{E}^{\mathbb{Q}[\bar{\mu}]}\left[e^{-\int_0^T (\varsigma(t,X_t)+\eta_0^* g_t)\,dt - \eta_1^* f_T}\right]}\,.$$

Barycentre of Models

Theorem 6

The constrained Barycentre drift is

$$\theta_{\eta^*}(t,x) = \bar{\boldsymbol{\mu}}(t,x) + \Sigma(t,x)\nabla \log \omega_{\eta^*}(t,x),$$

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where

$$\omega_{\eta^*}(t,x) = \mathbb{E}_{t,x}^{\mathbb{Q}[\bar{\boldsymbol{\mu}}]}\left[e^{-\int_t^T \frac{1}{2}\varsigma(u,X_u)\mathrm{d}u - \eta_0^* f(X_T) - \eta_1^* \int_0^T g(X_s)\mathrm{d}s} \right],$$

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Algorithm to simulate under $\mathbb{Q}^*$

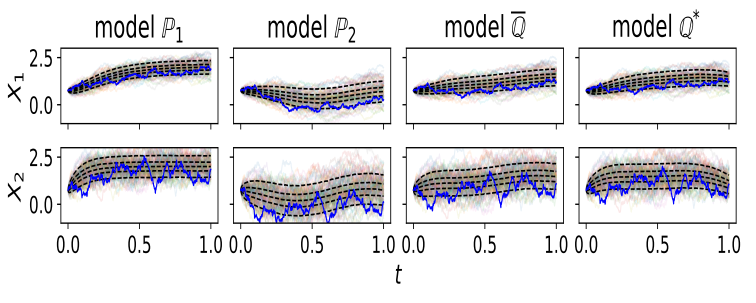
$i)$ calculate $\boldsymbol{\eta}^*$ by solving system of non-linear equations under $\mathbb{Q}[\bar{\boldsymbol{\mu}}]$

$ii)$ approx. ω with a NN and parameters $\mathfrak{o}$. By elicitability

$$\omega_{\eta^*}(\cdot,\cdot) = \arg\min_{\mathfrak{o}} \int_0^T \mathbb{E}^{\mathbb{Q}[\bar{\boldsymbol{\mu}}]}\left[\left(\omega_\eta(t,X_t)[\mathfrak{o}] - e^{-\int_t^T \frac{1}{2}\varsigma(u,X_u)+g(X_u)\mathrm{d}u-\eta^* f(X_T)}\right)^2\right]\mathrm{d}t$$

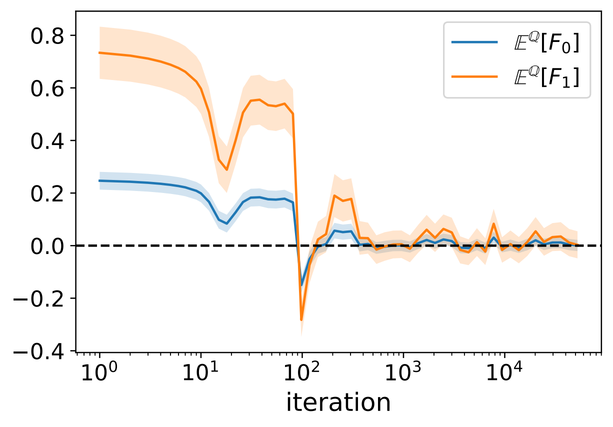
Bivariate constrained Barycentre

Constraints: $\mathbb{E}^{\mathbb{Q}}\big[X_{1,T} + X_{2,T}\big] = 1.2$, and $\text{var}^{\mathbb{Q}}\big[X_{1,T} + X_{2,T}\big] = 0.05$.



Convergence results of constraints

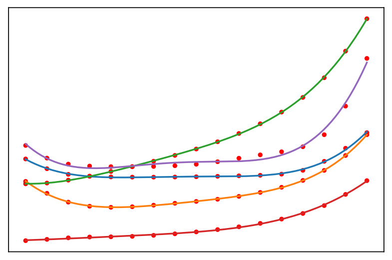
$$F_0 := X_{1,T} + X_{2,T} - 1.2 \quad \& \quad F_1 := (X_{1,T} + X_{2,T})^2 - 1.2^2 - 0.05$$



Example: Implied Volatility Smiles

Apply our framework to combine implied volatility (IV) smiles models

- Value of σ as a function of strike price Δ, such that the BS option price matches market prices for fixed time-to-maturity τ;
- Typical IV smiles $\sigma_t(\Delta)$ as a function of strike price Δ for different days;



Δ

Example: Implied Volatility Smiles

$i)$ daily IVs at discrete $\Delta's$ for time horizon $[0, 3T]$

$ii)$ project raw data onto basis functions (normalised Legendre polynomials)

$$\sigma_t(\Delta) = \sum_{j=1}^{5} a_{t,j} \;\; \phi_j(\Delta)$$

Example: Implied Volatility Smiles

$i)$ daily IVs at discrete $\Delta' s$ for time horizon $[0, 3T]$

$ii)$ project raw data onto basis functions (normalised Legendre polynomials)

$$\sigma_t(\Delta) = \sum_{j=1}^{5} a_{t,j} \;\; \phi_j(\Delta)$$

$iii)$ learn neural SDE for the dynamics of the coefficients $\boldsymbol{a}_t := (a_{t,1}, \ldots, a_{t,5})$

$$\mathrm{d}\boldsymbol{a}_t = \mu(\boldsymbol{a}_t)\, dt + \sigma(\boldsymbol{a}_t)\mathrm{d}W_t$$

Example: Implied Volatility Smiles

$i)$ daily IVs at discrete $\Delta's$ for time horizon $[0, 3T]$

$ii)$ project raw data onto basis functions (normalised Legendre polynomials)

$$\sigma_t(\Delta) = \sum_{j=1}^{5} \underbrace{a_{t,j}}_{X_{t,j}} \; \phi_j(\Delta) = f\big(X_{t,1}, \ldots, X_{t,5}\big)$$

$iii)$ learn neural SDE for the dynamics of the coefficients $\boldsymbol{a}_t := (a_{t,1}, \ldots, a_{t,5})$

$$\mathrm{d}\boldsymbol{a}_t = \mu(\boldsymbol{a}_t)\, dt + \sigma(\boldsymbol{a}_t)\mathrm{d}W_t$$

Example: Implied Volatility Smiles Expert's Models

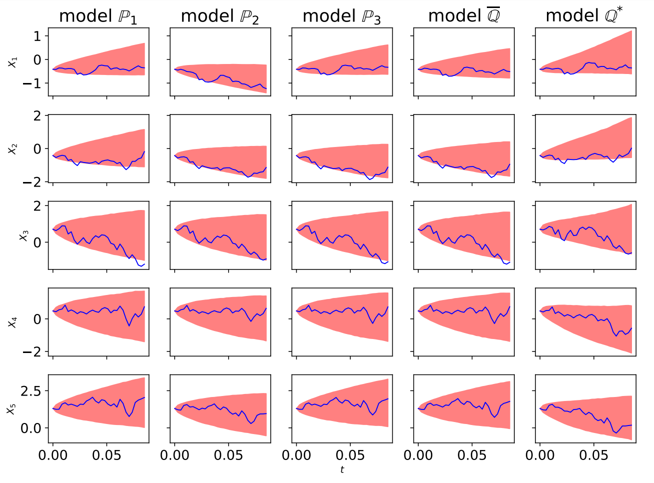
The 3 experts models:

$a)$ split data into 3 patches $\big[\,(k-1)\,T,\ k\,T\,\big)$, $k=1,2,3$

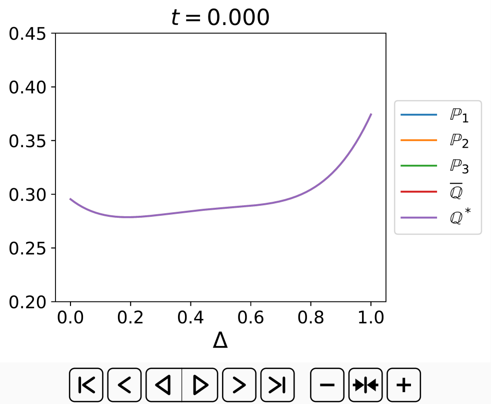
$b)$ Learn experts k's drift by fitting a neural SDE

$$\mathrm{d}\boldsymbol{a}_t = \mu^{(k)}(\boldsymbol{a}_t)\,dt + \sigma(\boldsymbol{a}_t)\,\mathrm{d}W_t^{(k)}\,, \qquad t\in\big[\,(k-1)\,T,\ k\,T\,\big)$$

skewness constraint: $\mathbb{E}^{\mathbb{Q}}\left[\partial_\Delta \sigma_t(\Delta)\big|_{\Delta=\frac{1}{2}}\right] = 0.05$



Barycentre IV Smiles II



Contributions

- Incorporating constraints / views into Lev́y-Itô processes
    - Analytical representation and algorithm
    - Application to stress testing

- Barycentres for diffusive models
    - Efficient deep learning algorithms to combine experts models
    - Application to Implied volatility smiles

Thank you!

https://pesenti.utstat.utoronto.ca/

links to publications:

Stress testing in insurance; IME

Modifying stochastic processes; SICON

Barycentres of stochastic processes; SIFIN