Extremile regression

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Any quantile is a median, any extremile is a mean

Focus of this presentation

Study a class of L^2-based risk measures which are comonotonically additive and expectations over the whole of the distribution.

Assume that F is continuous and strictly increasing. Then the quantile q_τ of F satisfies

$$[F(q_\tau)]^{\log(1/2)/\log(\tau)} = \tau^{\log(1/2)/\log(\tau)} = \exp(\log(1/2)) = 1/2.$$

So q_τ is the median of Z_τ, where Z_τ has c.d.f. $z \mapsto [F(z)]^{\log(1/2)/\log(\tau)}$.

Let $r(\tau) = \log(1/2)/\log(\tau)$ and $K_\tau(t) = t^{r(\tau)}$.

Definition (Extremile)

The extremile of order τ of Y is the expectation of $Z_\tau = Z_\tau(Y)$:

$$\xi_\tau = \mathbb{E}(Z_\tau), \text{ with } Z_\tau \text{ having c.d.f. } K_\tau \circ F.$$

Alternative formulations of extremiles

When $r(\tau)$ is a positive integer, we then have

$$\xi_\tau = \mathbb{E}(\max(Y_1, \ldots, Y_{r(\tau)}))$$

where the Y_i are independent copies of Y.

Extremiles are coherent and comonotonically additive. They are not elicitable, but their formulation as a minimizer suggests they can be backtested with a natural methodology (work in progress).

For regression: If $J_\tau(t) = K'_\tau(t)$, it holds that

$$\xi_\tau = \arg\min_{\theta \in \mathbb{R}} \mathbb{E}(J_\tau(F(Y))[(Y-\theta)^2 - Y^2]).$$

So just like expectiles, extremiles are defined through an asymmetric least squares criterion, but with a different weighting scheme.

Extremile regression

Let $(Y, X) \in \mathbb{R} \times \mathbb{R}^d$ be a response-covariate pair. Assume that the conditional c.d.f. $F(\cdot|x)$ is continuous. The τth extremile of this c.d.f. defines the τth regression extremile of Y given $X = x$.

Definition (Regression extremile)

The τth regression extremile of Y given $X = x$ is

$$\xi_\tau(x) = \arg\min_{\theta \in \mathbb{R}} \mathbb{E}(J_\tau(F(Y|x))[(Y - \theta)^2 - Y^2] \,|\, X = x).$$

This definition requires $\mathbb{E}(|Y| \,|\, X = x) < \infty$ to make sense.

Regression extremiles keep the interpretation of extremiles (expectation of maxima...) but applied to the conditional distribution instead.

Assume that a random sample (Y_i, X_i), $1 \le i \le n$ is available.

Estimation - extreme case

Focus now on the case $\tau = \tau_n \uparrow 1$ as $n \to \infty$. An extrapolation method is needed! Assume that

$$\forall y > 0, \lim_{t\to\infty} \frac{q_{1-(ty)^{-1}}(x)}{q_{1-t^{-1}}(x)} = y^{\gamma(x)}.$$

The conditional distribution has a Pareto-type tail with index $\gamma(x)$.

In this case, if $\mathbb{E}(\max(-Y,0)\,|\,X=x) < \infty$ and $0 < \gamma(x) < 1$ then

$$\frac{\xi_\tau(x)}{q_\tau(x)} \to \mathcal{G}(\gamma(x)) \text{ as } \tau \uparrow 1,$$

where $\mathcal{G}(s) = \Gamma(1-s)\{\log 2\}^s$ and Γ is Euler's Gamma function.

This means that

$$\widehat{\xi}_{\tau_n}(x) = \mathcal{G}(\widehat{\gamma}(x))\widehat{q}_{\tau_n}(x)$$

is an estimator of the extreme regression extremile $\xi_{\tau_n}(x)$.

Our choices? Focus on $d = 1$ for simplicity. We set first

$$\widehat{F}_{\mathrm{NW}}(y|x) = \sum_{i=1}^{n} \mathbb{1}\{Y_i \le y\} L\left(\frac{x - X_i}{h_n}\right) \Big/ \sum_{i=1}^{n} L\left(\frac{x - X_i}{h_n}\right).$$

Here L is a p.d.f. on $\mathbb{R}$. We then define a kernel estimator of $\widehat{\gamma}(x)$ as

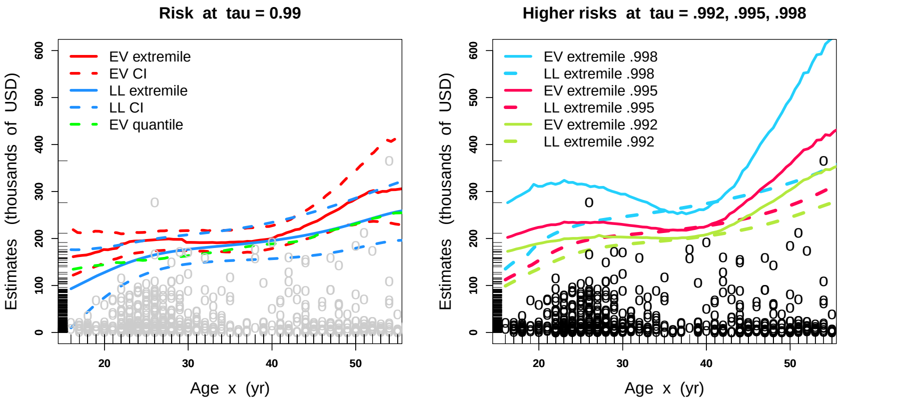
$$\widehat{\gamma}(x) = \frac{\sum_{j=1}^{J}\left[\log \widehat{F}_{\mathrm{NW}}^{-1}(1 - t_j(1 - a_n)|x) - \log \widehat{F}_{\mathrm{NW}}^{-1}(a_n|x)\right]}{\sum_{j=1}^{J} \log(1/t_j)}$$

where $1 = t_1 > t_2 > \cdots > t_J > 0$ are J weights. This is a kernel version of the (generalized) Pickands estimator. The estimator of $q_{\tau_n}(x)$ is then

$$\widehat{q}_{\tau_n}(x) = \left(\frac{1 - \tau_n}{1 - a_n}\right)^{-\widehat{\gamma}(x)} \widehat{F}_{\mathrm{NW}}^{-1}(a_n|x).$$

Here $a_n \uparrow 1$ with $nh_n(1 - a_n) \to \infty$: it is "extreme, but not too much".

For $t_j = 1/j$ the variance V_J is minimal for $J = 9$ with $V_9 \approx 1.25$.



Data set dataOhlsson (R package insuranceData) on $n = 670$ motorcycle-related claims recorded by the Swedish insurer Wasa.

Left: Estimates $\widehat{\xi}_{.99}(x)$ and $\widetilde{\xi}_{\mathrm{LL},.99}(x)$ (local linear estimator), corresponding 95% asymptotic confidence intervals, and $\widehat{q}_{.99}(x)$.

Right: Estimates $\widehat{\xi}_{\tau_n}(x)$ and $\widetilde{\xi}_{\mathrm{LL},\tau_n}(x)$ for $\tau_n = .992, .995, .998$.

Discussion

- An extremile above the mean is the mean of a distribution whose weight has been shifted to the right in a simple way.
- Extremiles are L^2 quantities, have various interpretations and closed forms, and do not rely solely on the tail event.
- They can be estimated at central and extreme levels using local linear estimation and semiparametric extrapolation.

Ongoing work and research perspectives? Forecast evaluation, dependent data (marginal/dynamic estimation)...

For (much!) more, see the following two papers:

Daouia, A., Gijbels, I., Stupfler, G. (2019). Extremiles: A new perspective on asymmetric least squares, *JASA* **114**(527): 1366–1381.

Daouia, A., Gijbels, I., Stupfler, G. (2021). Extremile regression, *JASA*, to appear.